Note: Particle Physics Seminar

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Abstract

Here is the brief summary of Particle physics seminar taught by Prof. Lingyun DAI. Enjoy the study !

Contents

1 SU(N) group

For SU(N) group, it's suppose that $UU^\dagger = \mathbf{1}$ and $\det U = \mathbf{1}$. For each component a_{ij} of the $U = (a_{ij})$, $a_{ij} = b_{ij} + ic_{ij}$, so it seems that there were $2n^2$ independent elements.

$$\sum_{k=1}^{N} a_{ik} \cdot a_{kj}^\dagger = \sum_{k=1}^{N} (b_{ik} + ic_{ik})(b_{jk} - ic_{jk}) = \sum_{k=1}^{N} (b_{ik}b_{jk} + c_{ik}c_{jk} + ic_{ik}b_{jk} - ib_{ik}c_{jk}) = \delta_{ij}$$

Extract the real part and the imaginary part,

$$\begin{cases} \sum_k (b_{ik}b_{jk} + c_{ik}c_{jk}) = 1 & (a) \\ \sum_k (c_{ik}b_{jk} - b_{ik}c_{jk}) = 0 & (b) \end{cases}$$

with two variables i and j varying from 1 to n. However, we find eqs from $i < j$ one-to-one equal to them of $i > j$, because of commutative multiplication. Thus we obtain $1 + 2 + \cdots + n - 1 = n(n-1)/2$ equations for both eq. (a) and eq. (b). If $i = j$, eq. (b) would always satisfy, but there are n independent eqs. from (a). Don't forget $\det U = 1$. Therefore, there are

$$2n^2 - \frac{n(n-1)}{2} \cdot 2 - n - 1 = n^2 - 1$$

independent elements in total, and 8 independent elements for SU(3) group.

Proposition 1. $u = \exp(iT_i\theta^i)$ is Hermitian.

$$uu^\dagger = \exp(iT_i\theta^i)\exp(-iT_i^\dagger\theta) = \exp\left[i(T_i - T_i^\dagger)\theta^i\right] = \mathbf{1} \qquad \Rightarrow \qquad T_i = T_i^\dagger$$

Proposition 2. T_i is traceless. (u is unitary matrix, so and $\det u = |u| = 1$)
Since the similar transformation would not change the trace of matrices, and as a Hermitian matrix, u has n independent eigenvectors, because

$$\ln|u| = \mathrm{tr}\,(\ln u) = \mathrm{tr}\left[\ln\exp(iT_i\theta^i)\right] = \mathrm{tr}(iT_i\theta^i) = 0$$

Question What is fundamental representation? - Defined in terms of matrices.

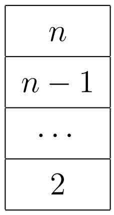
1.1 Young tableau & applications

Here is the simplest Young tableau, we say its order equal to n.

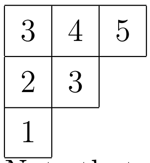
n

Define Hook length of each block = the number of blocks under it + right of it + 1 (itself).

Below is another Young tableau, with $(n-1)$ row but only 1 column.

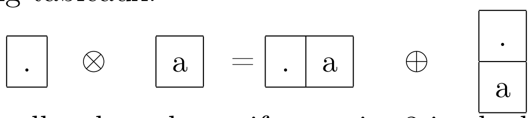


In general, the order of each Young tableau is the product of each component divided by all Hook lengths. Therefore, the order of the diagram above is equal to $\frac{n(n-1)\cdots 2}{(n-1)(n-2)\cdots 1} = n$.



The order of this Young tableau is equal to : $\frac{3\times4\times5\times2\times3\times1}{5\times3\times1\times3\times1\times1} = 8$

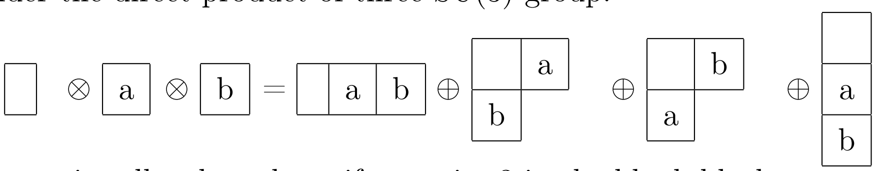
Note that apart from the last integer under the block, we should write next integer on the right of each block. Consider the direct product of two SU(3) group, we could describe it by fusing related Young tableaux.



We can easily write all orders above, if we write 3 in the blank block

$$3 \otimes 3 = 6 \oplus 3$$

Consider the direct product of three SU(3) group:



We can also write all orders above if we write 3 in the blank block:

$$3 \otimes 3 = 10 \oplus 8 \oplus 8 \oplus 1$$

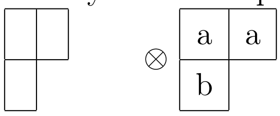
which means the baryons decuplet, octet, octet, singlet.

Note, there are some rules for more complex direct products:

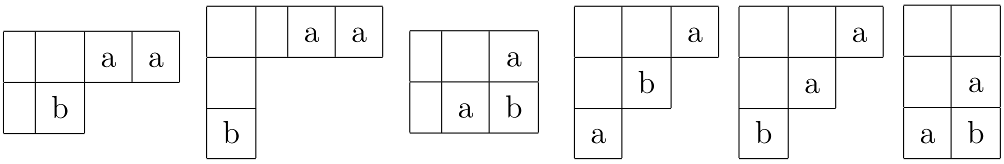
- focus on the right Young tableau;
- fill a in all first row, b the second, c the third ...
- add a blocks to the left Young tableau;
- don't put two a in the same column;
- after adding all a then add b blocks;
- from up to bottom, from right to left, the accumulated number of $b \leq$ that of a;

- of course, the column number should not increase from up to bottom;

Finally, let us try the direct product of two octet as an example, namely the equation below



and only these tableaux would admitted:



$$8 \otimes 8 = 27 \oplus 10 \oplus 10 \oplus 8 \oplus 8 \oplus 1$$

2 SU(3)

Suppose that u is a 3×3 matrix, with its determinant:

$$u_i^1 u_j^2 u_k^3 \epsilon^{ijk} = \det |u|$$

introduce Levi-Civita symbol

$$\epsilon^{abc} = u_i^a u_j^b u_k^c \epsilon^{ijk} = \epsilon^{abc} \cdot \det |u|$$

so ϵ^{abc} is irreducible representation.

For irreducible (n, m) tensor,

$$T^{i_1 i_2 \cdots}_{j_1 j_2 \cdots} \delta^{j_c}_{i_c} = 0$$

otherwise it contradicts with irreducibility.

For SU(n) group, $u = \exp(iT_i\theta_i), i = 1, 2, \cdots, n^2 - 1$, and $T_i = \lambda_i/2$. For SU(3) group, we have generators below:

$$\lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \qquad \lambda_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix} \qquad \lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$\lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix} \qquad \lambda_8 = \sqrt{\frac{1}{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix} \qquad \lambda_0 = \sqrt{\frac{2}{3}} \cdot \mathbf{1}$$

These Gell-Mann matrices satisfy $\lambda_a = \lambda_a^\dagger$, $\mathrm{Tr}(\lambda_a \lambda_b) = 2\delta_{ab}$, $\mathrm{Tr}(\lambda_a) = 0$, as well as:

$$\left[\frac{\lambda_a}{2}, \frac{\lambda_b}{2}\right] = i f_{abc} \frac{\lambda_c}{2} \qquad \text{or} \qquad [T_a, T_b] = i f_{abc} T_c$$

where f_{abc} is the coefficient:

$$f_{abc} = \frac{1}{4i}\mathrm{Tr}([\lambda_a, \lambda_b]\,\lambda_c)$$

abc	123	147	156	246	257	345	367	458	678
f_{abc}	1	1/2	−1/2	1/2	1/2	1/2	−1/2	$\sqrt{3}/2$	$\sqrt{3}/2$

In addition,

$$\{\lambda_i, \lambda_j\} = \frac{4}{3}\delta_{ij} + 2d_{ijk}\lambda_k$$

abc	118	146	157	228	247	256	338	344
d_{abc}	$1/\sqrt{3}$	1/2	1/2	$1/\sqrt{3}$	−1/2	1/2	$1/\sqrt{3}$	1/2
abc	355	366	377	448	558	668	778	888
d_{abc}	1/2	−1/2	−1/2	$-1/2\sqrt{3}$	$-1/2\sqrt{3}$	$-1/2\sqrt{3}$	$-1/2\sqrt{3}$	$-1/\sqrt{3}$

Introduce Casimir operator, also known as a Casimir invariant, which is a distinguished element of the center of the universal enveloping algebra of a Lie algebra.

$$C_2 = \sum_{i=1}^{8} T_i T_i = \frac{1}{2}\{I_+, I_-\} + I_3^2 + \frac{1}{2}\{U_+, U_-\} + \frac{1}{2}\{V_+, V_-\} + T_8^2 = \begin{pmatrix} 1/3 & 0 & 0 \\ 0 & 1/3 & 0 \\ 0 & 0 & 1/3 \end{pmatrix}$$

where (B number of baryons, S number of strangeness)

$$\begin{aligned} I_\pm &= T_1 \pm iT_2, & I_3 &= T_3 \\ U_\pm &= T_6 \pm iT_7, & U_3 &= \frac{\sqrt{3}}{2}T_8 - \frac{1}{2}T_3 = \frac{3}{4}Y - \frac{1}{2}I_3 \\ V_\pm &= T_4 \pm iT_5, & V_3 &= \frac{\sqrt{3}}{2}T_8 + \frac{1}{2}T_3 = \frac{3}{4}Y + \frac{1}{2}I_3 \\ Y &= \frac{2}{\sqrt{3}} = \frac{1}{2}(B+S) \end{aligned}$$

therefore $[C_2, T_i] = 0$; similarly, $[C_3, T_i] = 0$

$$C_3 = 2(d_{ijk} + if_{ijk})T_i T_j T_k$$

Easy to prove

$$[I_+, I_-] = 2I_3 \qquad [U_+, U_-] = 2U_3 \qquad [V_+, V_-] = 2V_3$$

(I_+, I_-, I_3) forms an SU(2) subgroup I; which is orthogonal to U(1) subgroup Y. (U_+, U_-, U_3) forms another SU(2) subgroup I; which is orthogonal to subgroup

$$Q = I_3 + \frac{Y}{2} = T_3 + \frac{1}{\sqrt{3}}T_8$$

I_3 for isospin, $Q =$ for charge, Y for hypercharge. For u-d-s three-flavor quarks,

$$I = \{1/2, -1/2, 0\} \qquad Q = \{2/3, -1/3, -1/3\} \qquad Y = \{1/3, 1/3, -2/3\}$$

In the isospin space, $I_+ |d\rangle \to |u\rangle$, $I_- |u\rangle \to |d\rangle$:

$$u = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \qquad d = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \qquad s = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$I_+ = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad I_- = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Similarly, $U_\pm$ could exchange d and s, $V_\pm$ could exchange s and u. Suppose $q^i = (u, d, s)$ a 3-vector, transforming like $q^i \to u^i_j q^j$ under SU(3); $\bar{q}_i = (\bar{u}, \bar{d}, \bar{s})$ is also a 3-vector.

For mesons and baryons, wave functions include

$$\Psi = \psi_{\text{space}} \cdot \psi_{\text{spin}} \cdot \psi_{\text{flavor}} \cdot \psi_{\text{color}}$$

2.1 Mesons & Baryons

Mesons are hadronic subatomic particles composed of an equal number of quarks and antiquarks, denoted by $q\bar{q} = 3 \times 3^*$.

$$\begin{array}{|c|} \hline 3 \\ \hline 2 \\ \hline \end{array} \otimes \begin{array}{|c|} \hline 3 \\ \hline \end{array} = \begin{array}{|c|} \hline 3 \\ \hline 2 \\ \hline 1 \\ \hline \end{array} \oplus \begin{array}{|c|c|} \hline 3 & 4 \\ \hline 2 & \\ \cline{1-1} \end{array}$$

$$\underline{3^*} \otimes 3 = 1 \oplus \underline{8}$$

Suppose

$$M'^i_j = q^i \bar{q}_j = \begin{pmatrix} u\bar{u} & u\bar{d} & u\bar{s} \\ d\bar{u} & d\bar{d} & d\bar{s} \\ s\bar{u} & s\bar{d} & s\bar{s} \end{pmatrix}$$

Since $M'^i_i \neq 0$, it's reducible tensor. Define another new traceless tensor

$$M^i_j = q^i \bar{q}_j - \frac{1}{3}\delta^i_j q^k \bar{q}_k = \begin{pmatrix} \frac{2u\bar{u} - d\bar{d} - s\bar{s}}{3} & u\bar{d} & u\bar{s} \\ d\bar{u} & \frac{2d\bar{d} - u\bar{u} - s\bar{s}}{3} & d\bar{s} \\ s\bar{u} & s\bar{d} & \frac{2s\bar{s} - u\bar{u} - d\bar{d}}{3} \end{pmatrix} = \begin{pmatrix} \frac{\pi^0}{\sqrt{2}} + \frac{\eta_8}{\sqrt{6}} & \pi^+ & K^+ \\ \pi^- & -\frac{\pi^0}{\sqrt{2}} + \frac{\eta_8}{\sqrt{6}} & K^0 \\ K^- & \bar{K}^0 & -\frac{2\eta_8}{\sqrt{6}} \end{pmatrix}$$

Then for baryons

$$\begin{array}{|c|} \hline 3 \\ \hline \end{array} \otimes \begin{array}{|c|} \hline 3 \\ \hline \end{array} \otimes \begin{array}{|c|} \hline 3 \\ \hline \end{array} = \begin{array}{|c|c|c|} \hline 3 & 4 & 5 \\ \hline \end{array} \oplus \begin{array}{|c|c|} \hline 3 & 4 \\ \hline 2 & \\ \cline{1-1} \end{array} \oplus \begin{array}{|c|c|} \hline 3 & 4 \\ \hline 2 & \\ \cline{1-1} \end{array} \oplus \begin{array}{|c|} \hline 3 \\ \hline 2 \\ \hline 1 \\ \hline \end{array}$$

3 QED

$$\vec{E} = -\vec{\nabla}\varphi - \frac{\partial \vec{A}}{\partial t} \qquad \vec{B} = \vec{\nabla} \times \vec{A}$$

Hamiltonian for QED with interaction:

$$H = \sqrt{(p - eQA)^2 + m^2} + Qe\phi$$

satisfying (q_i general coordinates, p_i regular momenta)

$$\frac{dq_i}{dt} = \frac{\partial H}{\partial p_i} \qquad \frac{dp_i}{dt} = -\frac{\partial H}{\partial q_i}$$

under non-relativistic limit,

$$H \approx \frac{(p - eQA)^2}{2m}$$

Easy to derive Lorentz force of charged particles in the electromagnetic field

$$\vec{F} = Qe\left(\vec{E} + \vec{v} \times \vec{B}\right)$$

so in order to consider interactions between particles and electromagnetic field, we only need

$$\vec{p} \to \vec{p} - Qe\vec{A} \qquad H \to H - Qe\varphi$$

$$\vec{p} - Qe\vec{A} \to -i\vec{\nabla} - Qe\vec{A} \qquad H - Qe\varphi \to i\frac{\partial}{\partial t} - Qe\varphi$$

by 4-vec $A^\mu = \left(\varphi, \vec{A}\right)$, we could write

$$i\partial^\mu - QeA^\mu = i(\partial^\mu + iQeA^\mu)$$

$$\partial_\mu \to \partial_\mu + iQeA_\mu$$

Therefore, for Lagrangian of free electrons,

$$\mathcal{L}_{\text{free}} = \bar{\psi}(i\gamma^\mu\partial_\mu - m)\psi$$

it turns to

$$\mathcal{L} = \bar{\psi}i\gamma^\mu(\partial_\mu + iQeA_\mu)\psi - m\bar{\psi}\psi$$

which is just the Lagrangian with interaction in QED.

$$\mathcal{L}_{\text{QED}} = \bar{\psi}(i\gamma^\mu\partial_\mu - m)\psi \qquad \Rightarrow \qquad \mathcal{L} = \bar{\psi}i\gamma^\mu(\partial_\mu + iQeA_\mu)\psi - m\bar{\psi}\psi = \bar{\psi}(i\not{D} - m)\psi$$

To keep QED global invariant, we add the transformation for ψ in the local Lagrangian

$$\psi \to \psi' = \exp\left(-iQe\theta\right)\psi$$

with arbitrary angle θ.

4 Weak interaction

Beta decays can be classified according to the angular momentum (L value) and total spin (S value) of the emitted radiation. Since total angular momentum must be conserved, including orbital and spin angular momentum, beta decay occurs by a variety of quantum state transitions to various nuclear angular momentum or spin states, namely Fermi transition ($\Delta S = 0$)

$$O^{14} \to N^{14} + e^+ + \nu_e$$

and Gamow-Teller transition ($\Delta S = 1$).

$$He^6 \to Li^6 + e^- + \bar{\nu}_e$$

Fermi posed effective Lagrangian

$$H = \frac{G}{\sqrt{2}} \left(\bar{\psi}_p \gamma_\mu \psi_n\right) \left(\bar{\psi}_e \gamma_\mu \psi_\nu\right)$$

where $J_\mu = \bar{\psi}_p \gamma_\mu \psi_n$ is hadronic current, $j_\mu = \bar{\psi}_e \gamma_\mu \psi_\nu$ is leptonic current. Note that for Gamow-Teller transition, we need axial-vector current:

$$J_\mu = \bar{\psi}_p \gamma_\mu \gamma_5 \psi_n$$

In 1963, N. Cabibbo introduced mixing currents (θ_C is Cabibbo angle)

$$J_\mu = J_\mu^{\Delta S=0} \cos\theta_C + J_\mu^{\Delta S=1} \sin\theta_C$$

Weak interaction could be classified according to initial/final states:

- all leptons for initial and final states

$$\tau^- \to \mu^- + \bar{\nu}_\mu + \nu_\tau$$

- both lepton and baryon exist

$$n \to p + e^- + \bar{\nu}_e$$

- all baryons for initial and final states

$$K^+ \to \pi^+ + \pi^0$$

for quarks, it writes

$$J_\mu = \bar{u}\gamma_\mu(1 - \gamma_5)(d\cos\theta_C + s\sin\theta_C)$$

4.1 V-A theory

Generally, for current-current coupling:

$$H = \sum_i \frac{G_i}{\sqrt{2}} \bar{\psi}_A \Gamma_i \psi_B \bar{\psi}_C \Gamma_i \psi \bar{D}$$

with five different Γ_i

1. S, scalar, $\Gamma_i = \mathbf{1}$;
2. V, vector, $\Gamma_i = \gamma_\mu$;
3. T, tensor, $\Gamma_i = \sigma_{\mu\nu} = \frac{i}{2}(\gamma_\mu \gamma_\nu - \gamma_\nu \gamma_\mu)$;
4. A, axial-vec, $\Gamma_i = \gamma_\mu \gamma_5$;
5. P, pseudo-scalar, $\Gamma_i = \gamma_5$;

According to Feynman and Gell-Mann, only V and A currents work for weak interactions. Take $\mu^- \to e^- + \bar{\nu}_e + \bar{\mu}$ as an example, (with $\bar{\psi}_e$ denoted by e simply)

$$H_W = \frac{G_\mu}{\sqrt{2}} \bar{e} \gamma_\mu (1 - \gamma_5) \nu_e \bar{\nu}_\mu \gamma^\mu (1 - \gamma_5) \mu + \text{h.c.}$$

and easy to derive decay width (ignore m_e^2/m_μ^2)

$$\Gamma_\mu = \frac{1}{\tau_\mu} \approx \frac{G_\mu^2 m_\mu^5}{192\pi^3}$$

u,d,s are eigenstates for mass; u'd's' are eigenstates for weak interaction:

$$\begin{pmatrix} d' \\ s' \end{pmatrix} = \begin{pmatrix} \cos\theta_C & \sin\theta_C \\ -\sin\theta_C & \cos\theta_C \end{pmatrix} \begin{pmatrix} d \\ s \end{pmatrix}$$

In 1973, Kobayashi and Maskawa popularized quarks to three generations

$$\begin{pmatrix} d' \\ s' \\ b' \end{pmatrix} = V_{\text{CKM}} \begin{pmatrix} d \\ s \\ b \end{pmatrix} = \begin{pmatrix} V_{ud} & V_{us} & V_{ub} \\ V_{cd} & V_{cs} & V_{cb} \\ V_{td} & V_{ts} & V_{tb} \end{pmatrix} \begin{pmatrix} d \\ s \\ b \end{pmatrix}$$

CKM matrix, namely Cabibbo-Kobayashi-Maskawa matrix.